

IFoA, Staple Inn, London

# A stochastic implementation of the APCI model

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Institute  
and Faculty  
of Actuaries

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# 1 Contributors

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Actuarial  
Research Centre

Institute and Faculty  
of Actuaries

# 2 Background

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- CMI released new projection spreadsheet.
- Calibration is done by new APCI model.
- See Continuous Mortality Investigation [2017].

- CMI intended APCI model for calibrating deterministic targeting spreadsheet.
- Richards et al. [2017] show how to implement it as a fully stochastic model.

# 3 APCI model

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$$\log m_{x,y} = \alpha_x + \beta_x(y - \bar{y}) + \kappa_y + \gamma_{y-x} \quad (1)$$

### 3 Related models for $\log m_{x,y}$



$$\text{Age-Period} : \alpha_x + \kappa_y \quad (2)$$

$$\text{APC} : \alpha_x + \kappa_y + \gamma_{y-x} \quad (3)$$

$$\text{Lee-Carter} : \alpha_x + \beta_x \kappa_y \quad (4)$$

$$\text{APCI} : \alpha_x + \beta_x (y - \bar{y}) + \kappa_y + \gamma_{y-x} \quad (5)$$

# 3 Model relationships

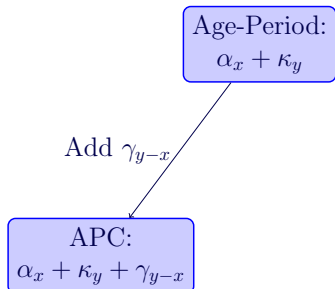
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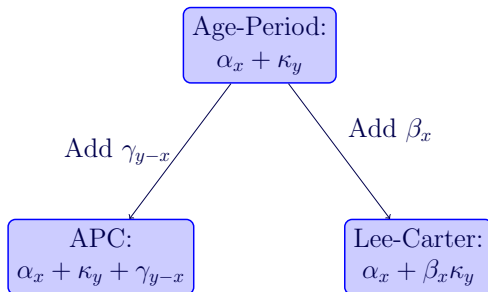
Age-Period:

$$\alpha_x + \kappa_y$$

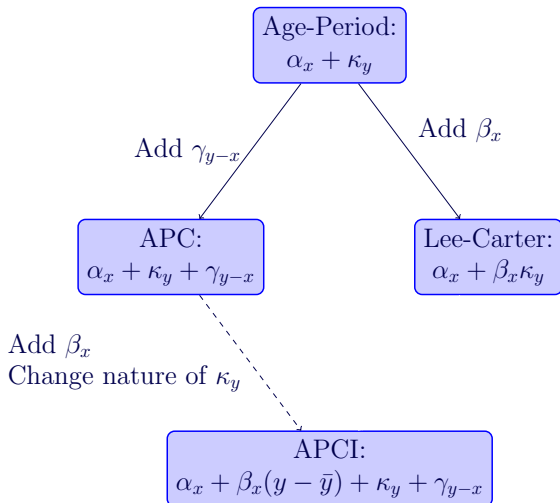
# 3 Model relationships



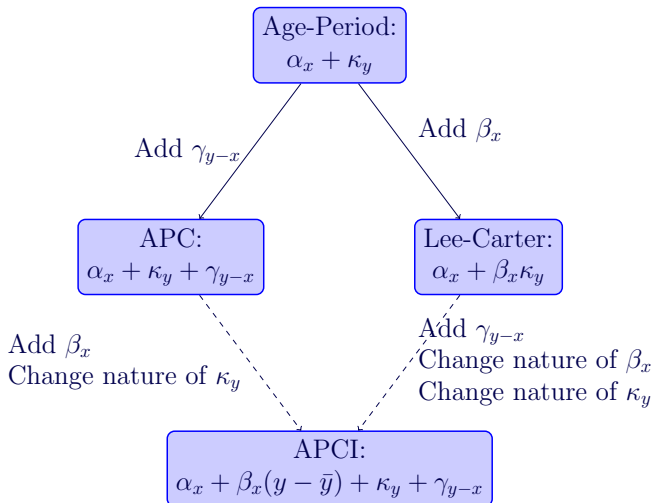
# 3 Model relationships



# 3 Model relationships



# 3 Model relationships



APCI model can be viewed superficially as either:

- An APC model with added Lee-Carter-like  $\beta_x$  term, or
- A Lee-Carter-like model with added  $\gamma_{y-x}$  cohort term.



...but there are important differences:

- In the Lee-Carter model the change in mortality is age-dependent:  $\beta_x \kappa_y$ .
- In the APCI model only the *expected* change is age-dependent:  $\beta_x (y - \bar{y})$ .
- $\kappa_y$  in the APCI model is very different to  $\kappa_y$  in the other models.

⇒ Although related to the APC or Lee-Carter models, the APCI model is not a generalization of either.

# 4 Fitting and constraints

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- All of these models have an infinite number of possible parameterisations.
- Pick the Age-Period model as a simple example. . .

If we set:

$$\alpha'_x = \alpha_x + v, \forall x$$

$$\kappa'_y = \kappa_y - v, \forall y$$

then the model will have the same fitted values for any real-valued  $v$ .



- Use an *identifiability constraint* to impose desired behaviour without changing fit.
- Choice of identifiability constraints helps interpretation and can make parameters like  $\kappa_y$  forecastable.

Age-Period model:

- Imposing  $\sum_y \kappa_y = 0$  does not change the fit...
- ...but it means that  $\alpha_x$  is (broadly) the average of  $\log \mu_{x,y}$  over the period.

$$\text{AP} : \sum_y \kappa_y = 0 \quad (6)$$

$$\text{LC} : \sum_y \kappa_y = 0, \sum_x \beta_x = 1 \quad (7)$$

$$\text{APC} : \sum_y \kappa_y = 0, \sum_{x,y} \gamma_c = 0, \sum_{x,y} (c - c_{\min} + 1) \gamma_c = 0 \quad (8)$$

where  $c = y - x$ .



APCI model uses five identifiability constraints:

$$\sum_y \kappa_y = 0 \quad (9)$$

$$\sum_y (y - y_1) \kappa_y = 0 \quad (10)$$

$$\sum_{x,y} \gamma_c = 0 \quad (11)$$

$$\sum_{x,y} (c - c_{\min} + 1) \gamma_c = 0 \quad (12)$$

$$\sum_{x,y} (c - c_{\min} + 1)^2 \gamma_c = 0 \quad (13)$$

- Continuous Mortality Investigation [2017] uses (for example)  $\sum_c \gamma_c = 0$ .  
 $\Rightarrow$  Cohort with one observation gets same weight as cohort with thirty observations?

- Cairns et al. [2009] weight according to number of observations, i.e.  $\sum_{x,y} \gamma_c = \sum_c w_c \gamma_c = 0$ .
- Cairns et al. [2009] approach preferable.
- See also Richards et al. [2017, Appendix C].

The Age-Period, APC and APCI models:

- are linear,
- use identifiability constraints, and
- have parameters that can be smoothed.

- Assume  $D_{x,y} \sim \text{Poisson}(E_{x,y}\mu_{x,y})$ .
- AP, APC and APCI models are penalized, smoothed GLMs.
- Lee-Carter model can fitted as pairwise conditional penalized, smoothed GLMs.

Currie [2013] sets out generalized GLM-fitting algorithm to:

- maximise likelihood,
- apply linear identifiability constraints, and
- smooth parameters.

Note that the Currie algorithm achieves these *simultaneously*, not in separate stages as in Continuous Mortality Investigation [2017].

- Identifiability constraints do not always have to be linear; see Girosi and King [2008], Cairns et al. [2009] and Richards and Currie [2009].
- However, *proving* that a constraint is an identifiability constraint is harder if it is non-linear.
- The Currie [2013] algorithm works with linear constraints only.

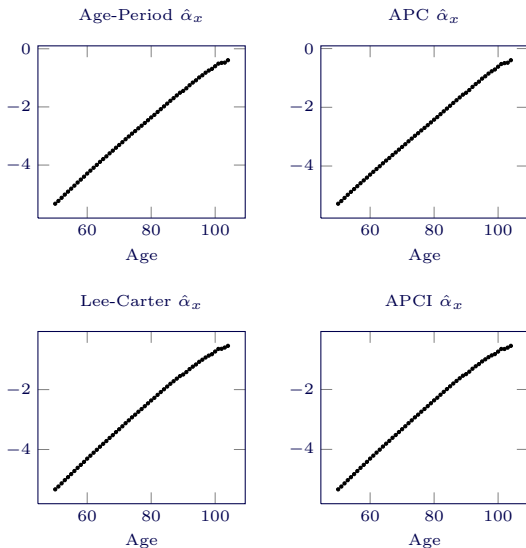
# 5 Parameter estimates

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Parameter estimates  $\hat{\alpha}_x$  for four unsmoothed models.

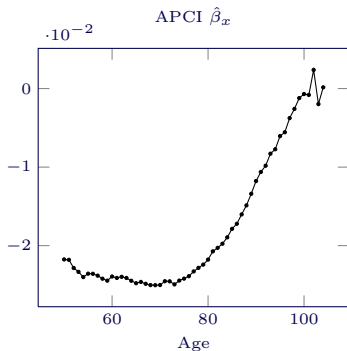
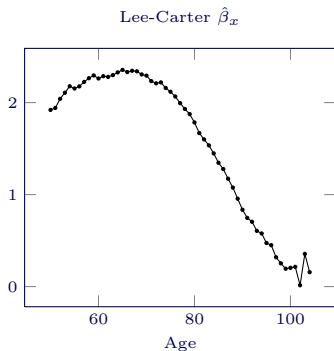


$\Rightarrow \alpha_x$  plays the same role across all four models,  
i.e. average log mortality by age.

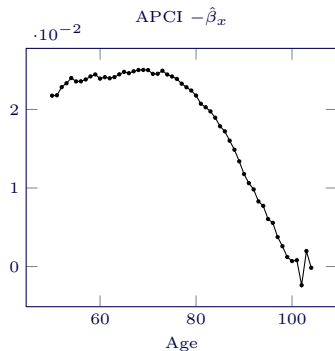
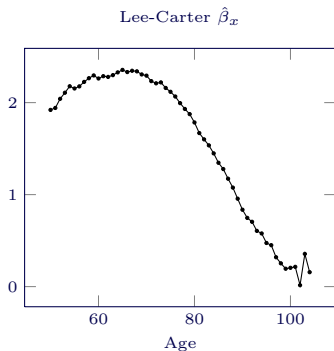
...as long as  $\sum_y \kappa_y = 0$ .

$\Rightarrow \alpha_x$  could be smoothed to reduce *effective dimension* of model.

Parameter estimates  $\hat{\beta}_x$  for Lee-Carter and APCI models (both unsmoothed).



Parameter estimates  $\hat{\beta}_x$  for Lee-Carter and  $-\hat{\beta}_x$  for APCI models (both unsmoothed).



- $\beta_x$  plays an analogous role in the Lee-Carter and APCI models, namely an age-related modulation of the time index.
- $\beta_x$  in APCI model operates on a quite different scale due to  $(y - \bar{y})$  term.
- $\beta_x$  in APCI model would be better multiplied by  $(\bar{y} - y)$  term...  
...and have a constraint on  $\beta_x$  analogous to the Lee-Carter one.

- Like  $\alpha_x$ ,  $\beta_x$  could be smoothed to reduce effective dimension of model.
- Smoothing  $\beta_x$  also improves forecasting properties; see Delwarde et al. [2007].

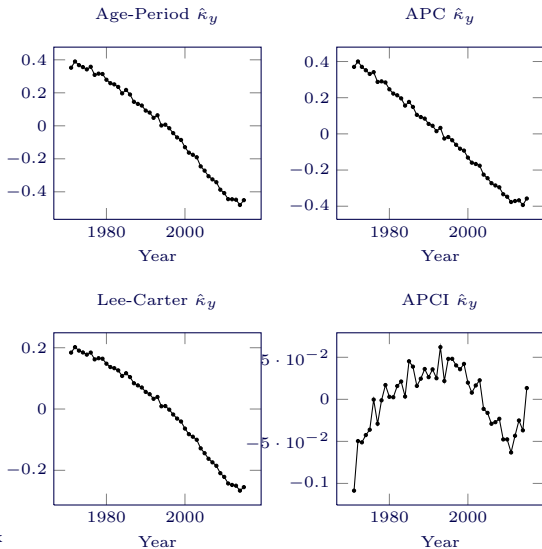
Note that the APCI model has *two* time-varying components:

1. An age-dependent central linear trend,  $(y - \bar{y})$ , and
2. An *unmodulated*, non-linear term,  $\kappa_y$ .

- $\alpha_x$  and  $\beta_x$  play similar roles across all models.
- What about  $\kappa_y$  and  $\gamma_{y-x}$ ?

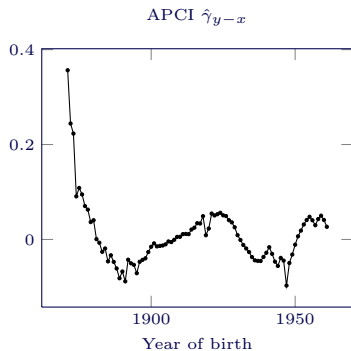
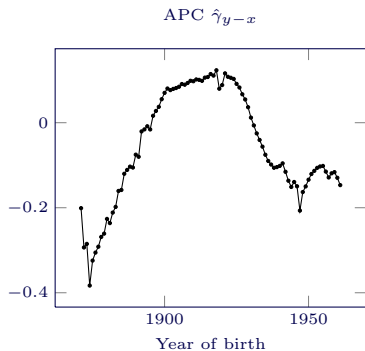


Parameter estimates  $\hat{\kappa}_y$  for four unsmoothed models.



- $\kappa_y$  plays a similar role in the Age-Period, APC and Lee-Carter models.
- $\kappa_y$  plays a very different role in the APCI model.
- APCI  $\hat{\kappa}_y$  values have less of a clear trend pattern for forecasting.
- APCI  $\hat{\kappa}_y$  values are strongly influenced by structural decisions made elsewhere in the model.

Parameter estimates  $\hat{\gamma}_{y-x}$  for APC and APCI models (both unsmoothed).



- The  $\gamma_{y-x}$  values appear to play analogous roles in the APC and APCI models...  
...yet the values taken and the shapes displayed are very different.
- If values and shapes are so different, what do  $\gamma_{y-x}$  values represent?
- $\gamma_{y-x}$  don't have an interpretation independent of the other parameters in the same model...  
... $\gamma_{y-x}$  don't describe cohort effects in any meaningful way.

# 6 Smoothing

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- Continuous Mortality Investigation [2017] smooths all parameters.
- However, only  $\alpha_x$  and  $\beta_x$  exhibit regular behaviour.
- Does it make sense to smooth  $\kappa_y$  and  $\gamma_{y-x}$ ?

- CMI's smoothing parameter for  $\kappa_y$  is  $S_\kappa$ .
- Smoothing penalty for  $\kappa_y$  is

$$10^{S_\kappa} \sum_{y=3}^{n_y} (\kappa_y - 2\kappa_{y-1} + \kappa_{y-2})^2.$$

- Value for  $S_\kappa$  is set subjectively.
- What is the impact of smoothing  $\kappa_y$ ?

*life expectancies are [...] very sensitive to the choice made for  $S_\kappa$ , with the impact varying across the age range. At ages above 45, changing  $S_\kappa$  by 1 has a greater impact than changing the long-term rate by 0.5%.”*

Continuous Mortality Investigation [2016, page 42]

See also <https://www.longevity.co.uk/site/informationmatrix/signalornoise.html>





- $S_\kappa$  has a large impact because  $\kappa_y$  collects features left over from other parts of the model structure.
- Indeed,  $\kappa_y$  collects every remaining period effect and applies it without any age modulation.
- If  $\kappa_y$  is a “left-over”, should one smooth it at all?

# 7 Value-at-Risk (VaR)

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*“Whereas a catastrophe can occur in an instant, longevity risk takes decades to unfold”*

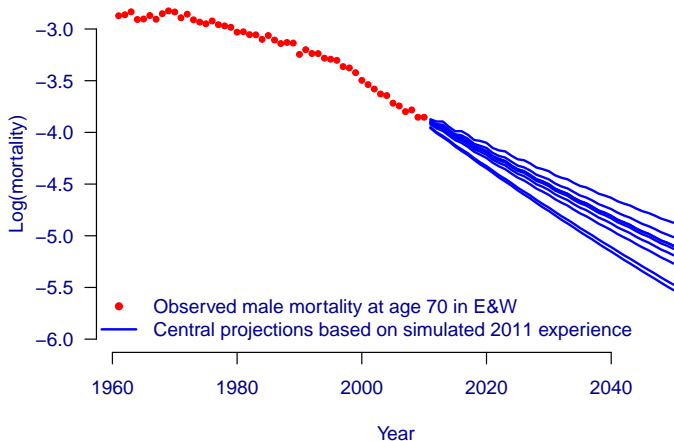
**The Economist [2012]**



Solution from Richards et al. [2014]:

- Simulate next year's experience.
- Refit the model.
- Value liabilities.
- Repeat...

# 7 Sensitivity of forecast



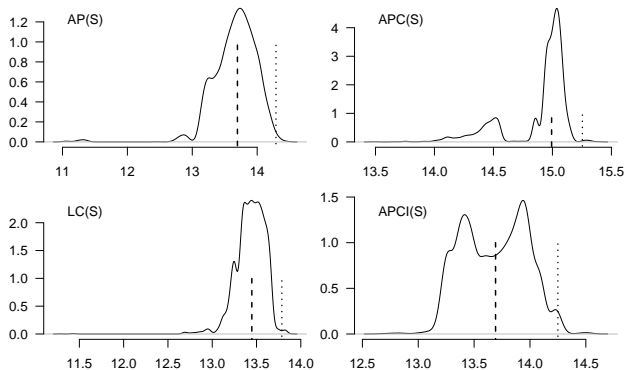
Approach from Kleinow and Richards [2016] for parameter uncertainty:

- $\gamma_{y-x}$ : use ARIMA model without mean.
- $\kappa_y$  under AP, APC and LC models: use ARIMA model with mean.
- $\kappa_y$  under APCI model: use ARIMA model *without* mean.

# 7 Liability densities



Value-at-risk capital requirements for annuities payable to male 70-year-olds. Source: Richards et al. [2017, Table 4].





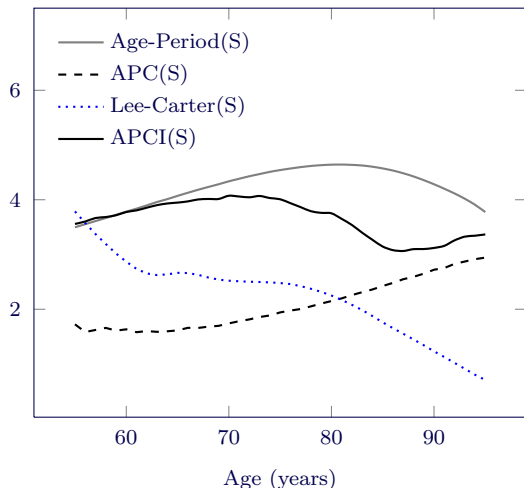
- Variety of density shapes.  
⇒ not all unimodal.
- Considerable variability between models.  
⇒ need to use multiple models.



# 7 Value-at-risk



VaR99.5% capital-requirement percentages by age for four models.  
Source: Richards et al. [2017].





Q. Why do capital requirements reduce with age for Lee-Carter, but not with APCI?

A.  $\kappa_y$  is unmodulated by age in APCI model.

# 8 Conclusions

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- APCI model is implementable as a fully stochastic model.
- APCI model shares features and drawbacks with Age-Period, APC and Lee-Carter models.
- Smoothing APCI  $\hat{\alpha}_x$  and  $\hat{\beta}_x$  seems sensible.
- Smoothing APCI  $\hat{\kappa}_y$  and  $\hat{\gamma}_{y-x}$  is not sensible.
- Currie [2013] algorithm makes fitting penalized, smoothed GLMs straightforward.

- A. J. G. Cairns, D. Blake, K. Dowd, G. D. Coughlan, D. Epstein, A. Ong, and I. Balevich. A quantitative comparison of stochastic mortality models using data from England and Wales and the United States. *North American Actuarial Journal*, 13(1):1–35, 2009.
- Continuous Mortality Investigation. *CMI Mortality Projections Model consultation — technical paper*. Working Paper 91, 2016.
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- I. D. Currie. Smoothing constrained generalized linear models with an application to the Lee-Carter model. *Statistical Modelling*, 13(1):69–93, 2013.
- A. Delwarde, M. Denuit, and P. H. C. Eilers. Smoothing the Lee-Carter and Poisson log-bilinear models for mortality forecasting: a penalized likelihood approach. *Statistical Modelling*, 7:29–48, 2007.
- F. Girosi and G. King. *Demographic Forecasting*. Princeton University Press, 2008. ISBN 978-0-691-13095-8.



- T. Kleinow and S. J. Richards. Parameter risk in time-series mortality forecasts. *Scandinavian Actuarial Journal*, 2016(10):1–25, 2016.
- S. J. Richards and I. D. Currie. Longevity risk and annuity pricing with the Lee-Carter model. *British Actuarial Journal*, 15(II) No. 65:317–365 (with discussion), 2009.
- S. J. Richards, I. D. Currie, and G. P. Ritchie. A value-at-risk framework for longevity trend risk. *British Actuarial Journal*, 19 (1):116–167, 2014.

S. J. Richards, I. D. Currie, T. Kleinow, and G. P. Ritchie. *A stochastic implementation of the APCI model for mortality projections*, 2017.

The Economist. The ferment of finance. *Special report on financial innovation*, February 25th 2012:8, 2012.

More on longevity risk at [www.longevity.co.uk](http://www.longevity.co.uk)

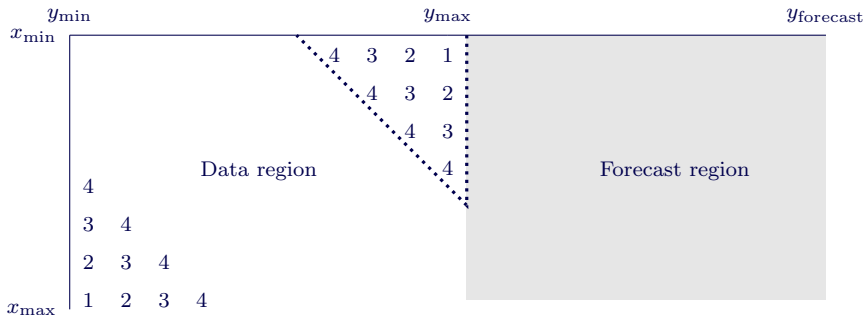


# 10 Constraints (again)

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Number of observations for each cohort in the data region.

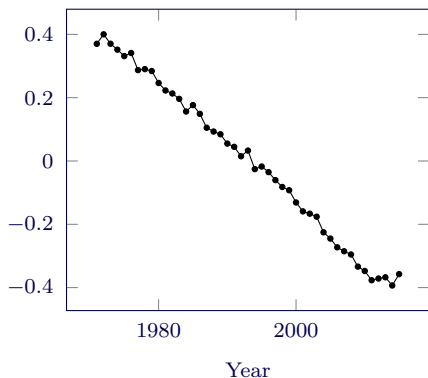


- Both Continuous Mortality Investigation [2017] and Richards et al. [2017] avoid estimating “corner cohorts”.
- This means not all constraints are required for identifiability.
- Continuous Mortality Investigation [2017] and Richards et al. [2017] both fit over-constrained APCI models.
- What impact does this have?

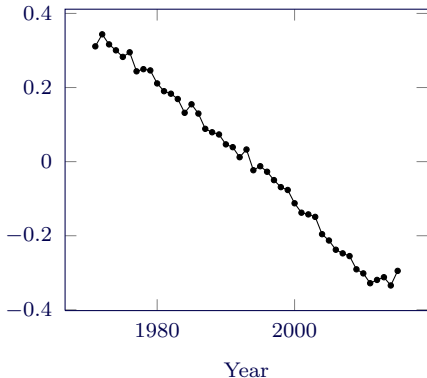
- Over-constrained models reduce the goodness-of-fit...  
...but can be used to impose desirable behaviour on parameters.

## Parameter estimates $\hat{\kappa}_y$ APC(S) model

$\hat{\kappa}_y$  (over-constrained)

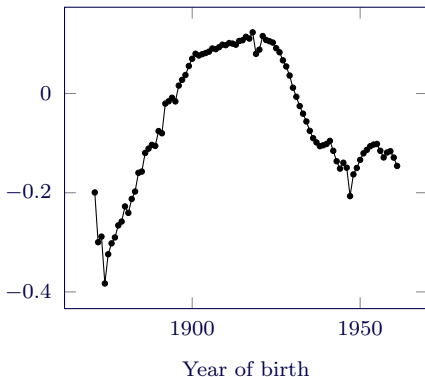


$\hat{\kappa}_y$  (minimal constraints)

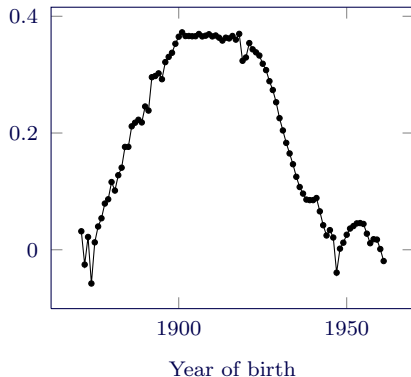


## Parameter estimates $\hat{\gamma}_{y-x}$ APC(S) model

$\hat{\gamma}_{y-x}$  (over-constrained)



$\hat{\gamma}_{y-x}$  (minimal constraints)

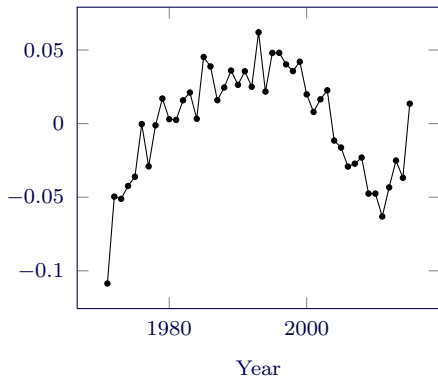




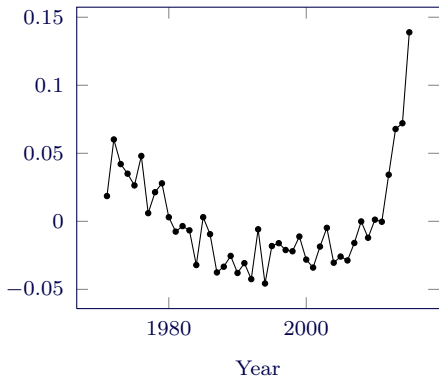
- $\hat{\kappa}_y$  robust to over-constrained model.
- Values for  $\hat{\gamma}_{y-x}$  differ, but shape similar.

## Parameter estimates $\hat{\kappa}_y$ APCI(S) model

$\hat{\kappa}_y$  (over-constrained)



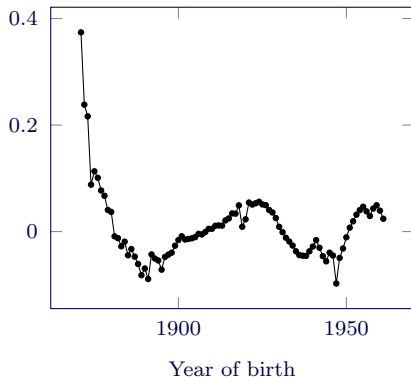
$\hat{\kappa}_y$  (minimal constraints)



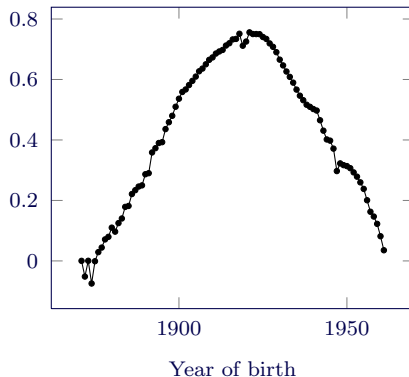


## Parameter estimates $\hat{\gamma}_{y-x}$ APCI(S) model

$\hat{\gamma}_{y-x}$  (over-constrained)



$\hat{\gamma}_{y-x}$  (minimal constraints)



- Neither  $\hat{\kappa}_y$  nor  $\hat{\gamma}_{y-x}$  robust to over-constrained model.
  - $\kappa_y$  in APCI model is a term which picks up left-over aspects of fit.
  - $\hat{\gamma}_{y-x}$  changes radically depending on constraint choices.
- ⇒ What are the implications for the CMI spreadsheet of using  $\hat{\gamma}_{y-x}$  from APCI model?