ICC, Birmingham

The APCI model — a stochastic implementation.

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Overview

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5. Parameter estimates
6. Smoothing
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8. Conclusions
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1 Contributors

1 Contributors

LONGEVITAS

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Actuarial Research Centre

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2 Background

2 Background

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- CMI released new projection spreadsheet.
- Calibration is done by new APCI model.
- See Continuous Mortality Investigation [2017].

2 Background

- CMI intended APCI model for calibrating deterministic targeting spreadsheet.
- Richards et al. [2017] show how to implement it as a fully stochastic model.
- Presented at sessional meeting of IFoA on 16th October 2017.
- Paper and materials at www.longevitas.co.uk/apci

3 APCI model

3 APCI model

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$$\log m_{x,y} = \alpha_x + \beta_x(y - \bar{y}) + \kappa_y + \gamma_{y-x} \qquad (1)$$

3 Related models for $\log m_{x,y}$

$$\text{Age-Period} : \alpha_x + \kappa_y \quad (2)$$

$$\text{APC} : \alpha_x + \kappa_y + \gamma_{y-x} \quad (3)$$

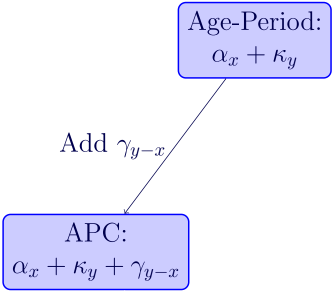
$$\text{Lee-Carter} : \alpha_x + \beta_x \kappa_y \quad (4)$$

$$\text{APCI} : \alpha_x + \beta_x (y - \bar{y}) + \kappa_y + \gamma_{y-x} \quad (5)$$

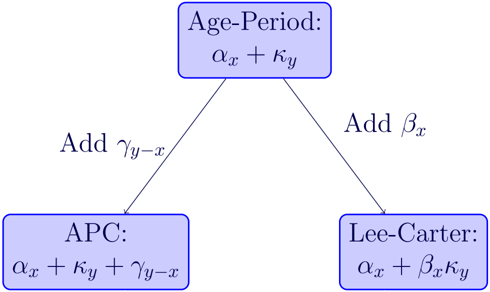
3 Model relationships

Age-Period:
$\alpha_x + \kappa_y$

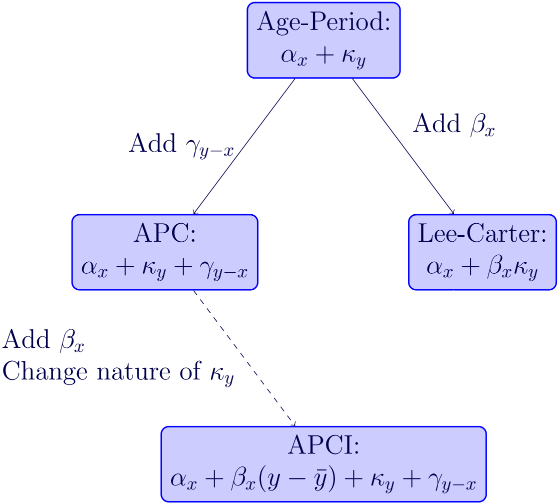
3 Model relationships



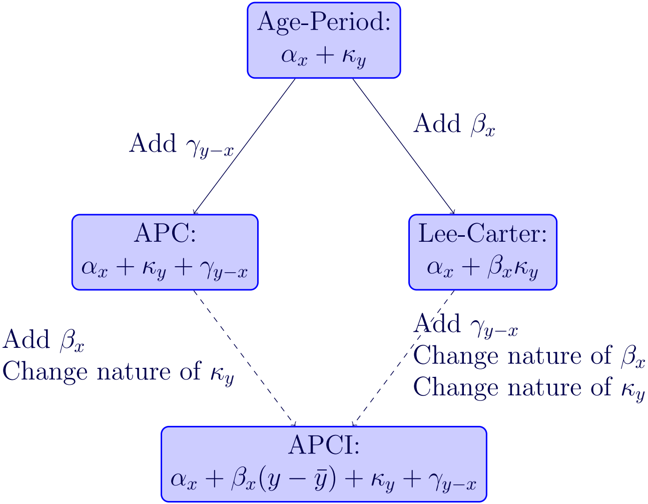
3 Model relationships



3 Model relationships



3 Model relationships



3 APCI model

APCI model can be viewed superficially as either:

- An APC model with added Lee-Carter-like β_x term, or
- A Lee-Carter-like model with added γ_{y-x} cohort term.

3 APCI model

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…but there are important differences:

- In the Lee-Carter model the change in mortality is age-dependent: $\beta_x \kappa_y$.
- In the APCI model only the *expected* change is age-dependent: $\beta_x(y - \bar{y})$.
- κ_y in the APCI model is very different to κ_y in the other models.

3 APCI model

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$\Rightarrow$ Although related to the APC or Lee-Carter models, the APCI model is not a generalization of either.

4 Fitting and constraints

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4 Identifiability

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- All of these models have an infinite number of possible parameterisations.
- Pick the Age-Period model as a simple example. . .

4 Identifiability — problem

If we set:

$$\alpha'_x = \alpha_x + v, \forall x$$
$$\kappa'_y = \kappa_y - v, \forall y$$

then the model will have the same fitted values for any real-valued v.

4 Identifiability — solution

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- Use an *identifiability constraint* to impose desired behaviour without changing fit.
- Choice of identifiability constraints helps interpretation and can make parameters like κ_y forecastable.

4 Constraints

Age-Period model:

- Imposing $\sum_y \kappa_y = 0$ does not change the fit...
- ...but it means that α_x is (broadly) the average of $\log \mu_{x,y}$ over the period.

4 Constraints used

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$$\text{AP}: \sum_y \kappa_y = 0 \tag{6}$$

$$\text{LC}: \sum_y \kappa_y = 0, \sum_x \beta_x = 1 \tag{7}$$

$$\text{APC}: \sum_y \kappa_y = 0, \sum_{x,y} \gamma_c = 0, \sum_{x,y} (c - c_{\min} + 1)\gamma_c = 0 \tag{8}$$

where $c = y - x$.

4 Constraints used

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APCI model uses five identifiability constraints:

$$\sum_y \kappa_y = 0 \tag{9}$$

$$\sum_y (y - y_1)\kappa_y = 0 \tag{10}$$

$$\sum_{x,y} \gamma_c = 0 \tag{11}$$

$$\sum_{x,y} (c - c_{\min} + 1)\gamma_c = 0 \tag{12}$$

$$\sum_{x,y} (c - c_{\min} + 1)^2\gamma_c = 0 \tag{13}$$

4 Not all cohorts are equal

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- Continuous Mortality Investigation [2017] uses (for example) $\sum_c \gamma_c = 0$.
  $\Rightarrow$ Cohort with one observation gets same weight as cohort with thirty observations?

4 Not all cohorts are equal

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- Cairns et al. [2009] weight according to number of observations, i.e. $\sum_{x,y} \gamma_c = \sum_c w_c \gamma_c = 0$.
- Cairns et al. [2009] approach preferable.
- See also Richards et al. [2017, Appendix C].

The Age-Period, APC and APCI models:

- are linear,
- use identifiability constraints, and
- have parameters that can be smoothed.

4 Fitting

- Assume $D_{x,y} \sim \text{Poisson}(E_{x,y}\mu_{x,y})$.
- AP, APC and APCI models are penalized, smoothed GLMs.
- Lee-Carter model can fitted as pairwise conditional penalized, smoothed GLMs.

4 Fitting

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Currie [2013] sets out generalized GLM-fitting algorithm to:

- maximise likelihood,
- apply linear identifiability constraints, and
- smooth parameters.

Note that the Currie algorithm achieves these *simultaneously*, not in separate stages as in Continuous Mortality Investigation [2017].

4 Constraints

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- Identifiability constraints do not always have to be linear; see Girosi and King [2008], Cairns et al. [2009] and Richards and Currie [2009].
- However, *proving* that a constraint is an identifiability constraint is harder if it is non-linear.
- The Currie [2013] algorithm works with linear constraints only.

5 Parameter estimates

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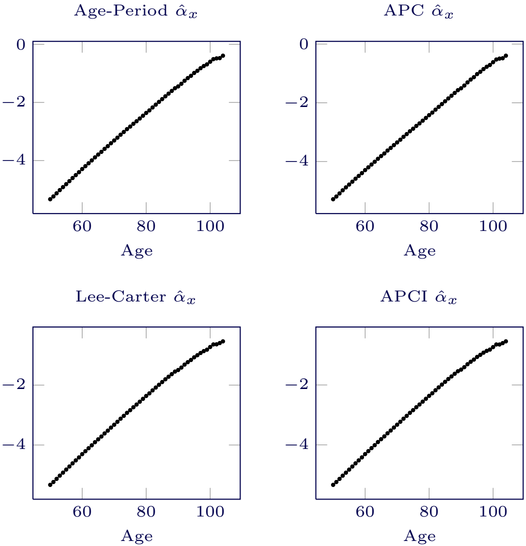
Parameter estimates $\hat{\alpha}_x$ for four unsmoothed models.



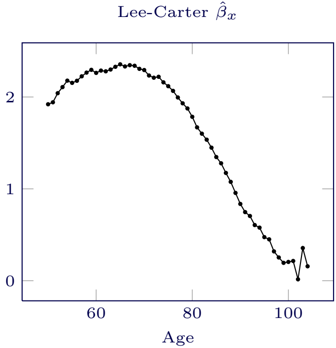
5 α_x

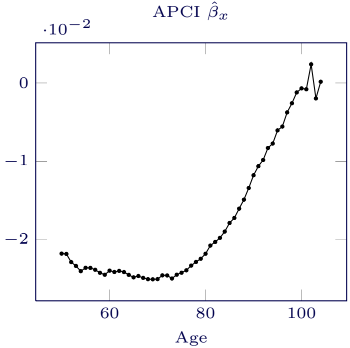
$\Rightarrow \alpha_x$ plays the same role across all four models, i.e. average log mortality by age.

...as long as $\sum_y \kappa_y = 0$.

$\Rightarrow \alpha_x$ could be smoothed to reduce *effective dimension* of model.

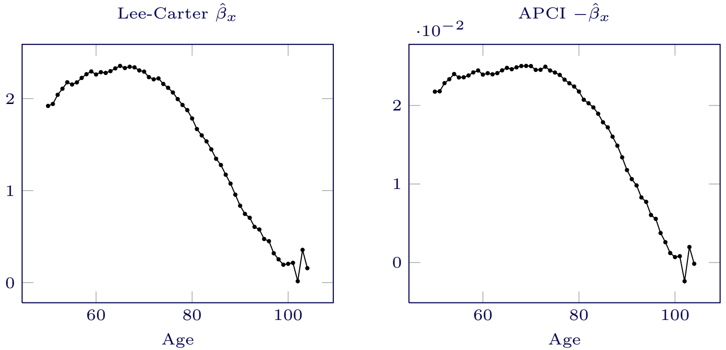
Parameter estimates $\hat{\beta}_x$ for Lee-Carter and APCI models (both unsmoothed).





5 β_x

Parameter estimates $\hat{\beta}_x$ for Lee-Carter and $-\hat{\beta}_x$ for APCI models (both unsmoothed).



5 β_x

- β_x plays an analogous role in the Lee-Carter and APCI models, namely an age-related modulation of the time index.
- β_x in APCI model operates on a quite different scale due to $(y - \bar{y})$ term.
- β_x in APCI model would be better multiplied by $(\bar{y} - y)$ term...

  ...and have a constraint on β_x analogous to the Lee-Carter one.

5 β_x

- Like α_x, β_x could be smoothed to reduce effective dimension of model.
- Smoothing β_x also improves forecasting properties; see Delwarde et al. [2007].

5 β_x

Note that the APCI model has *two* time-varying components:

1. An age-dependent central linear trend, $(y - \bar{y})$, and
2. An *unmodulated*, non-linear term, κ_y.

5 Conclusions for α_x and β_x

- α_x and β_x play similar roles across all models.
- What about κ_y and γ_{y-x}?

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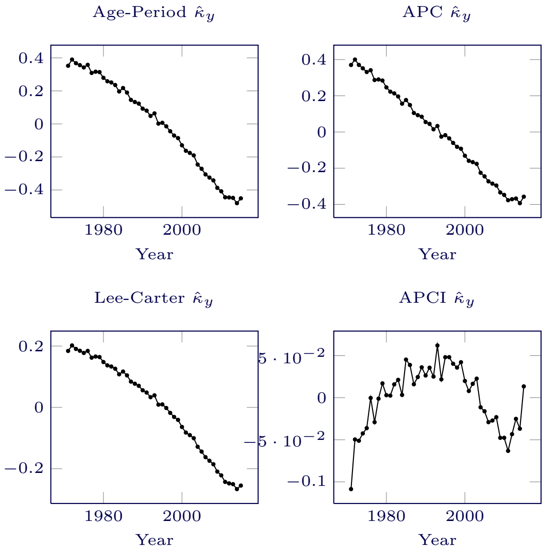
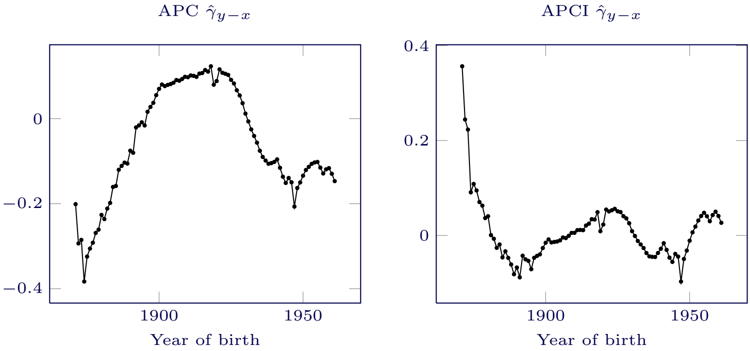
Parameter estimates $\hat{\kappa}_y$ for four unsmoothed models.



5 κ_y

- κ_y plays a similar role in the Age-Period, APC and Lee-Carter models.
- κ_y plays a very different role in the APCI model.
- APCI $\hat{\kappa}_y$ values have less of a clear trend pattern for forecasting.
- APCI $\hat{\kappa}_y$ values are strongly influenced by structural decisions made elsewhere in the model.

Parameter estimates $\hat{\gamma}_{y-x}$ for APC and APCI models (both unsmoothed).



5 γ_{y-x}

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- The γ_{y-x} values appear to play analogous roles in the APC and APCI models…

  ….yet the values taken and the shapes displayed are very different.
- If values and shapes are so different, what do γ_{y-x} values represent?
- γ_{y-x} don't have an interpretation independent of the other parameters in the same model…

  … γ_{y-x} don't describe cohort effects in any meaningful way.

6 To smooth or not to smooth?

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- Continuous Mortality Investigation [2017] smooths all parameters.
- However, only α_x and β_x exhibit regular behaviour.
- Does it make sense to smooth κ_y and γ_{y-x}?

6 To smooth or not to smooth?

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- CMI's smoothing parameter for κ_y is S_κ.
- Smoothing penalty for κ_y is
  $$10^{S_\kappa}\sum_{y=3}^{n_y}(\kappa_y - 2\kappa_{y-1} + \kappa_{y-2})^2.$$
- Value for S_κ is set subjectively.
- What is the impact of smoothing κ_y?

6 Impact of smoothing APCI κ_y

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life expectancies are [...] very sensitive to the choice made for S_κ, with the impact varying across the age range. At ages above 45, changing S_κ by 1 has a greater impact than changing the long-term rate by 0.5%."

Continuous Mortality Investigation [2016, page 42]

See also https://www.longevitas.co.uk/site/informationmatrix/signalornoise.html

6 Impact of smoothing APCI κ_y

- S_κ has a large impact because κ_y collects features left over from other parts of the model structure.
- Indeed, κ_y collects every remaining period effect and applies it without any age modulation.
- If κ_y is a “left-over”, should one smooth it at all?

7 Value-at-Risk (VaR)

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7 Trend risk v. one-year view?

"Whereas a catastrophe can occur in an instant, longevity risk takes decades to unfold"

The Economist [2012]

7 Trend risk v. one-year view

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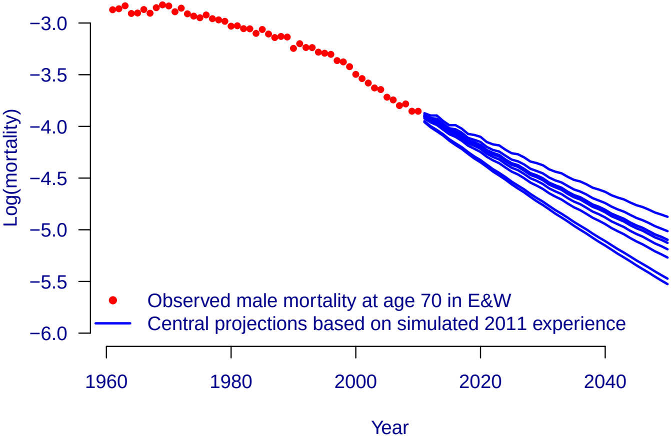
Solution from Richards et al. [2014]:

- Simulate next year's experience.
- Refit the model.
- Value liabilities.
- Repeat...

7 Sensitivity of forecast



Source: Lee-Carter example from Richards et al. [2014].

7 Forecasting

Approach from Kleinow and Richards [2016] for parameter uncertainty:

- γ_{y-x}: use ARIMA model without mean.
- κ_y under AP, APC and LC models: use ARIMA model with mean.
- κ_y under APCI model: use ARIMA model *without* mean.

7 Liability densities

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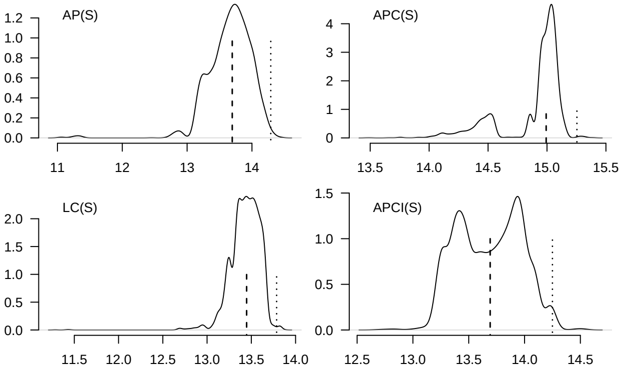
Value-at-risk capital requirements for annuities payable to male 70-year-olds. Source: Richards et al. [2017, Table 4].



See also https://www.longevitas.co.uk/site/informationmatrix/twinpeaks.html

7 Value-at-risk

- Variety of density shapes.
  $\Rightarrow$ not all unimodal.
- Considerable variability between models.
  $\Rightarrow$ need to use multiple models.

7 Value-at-risk

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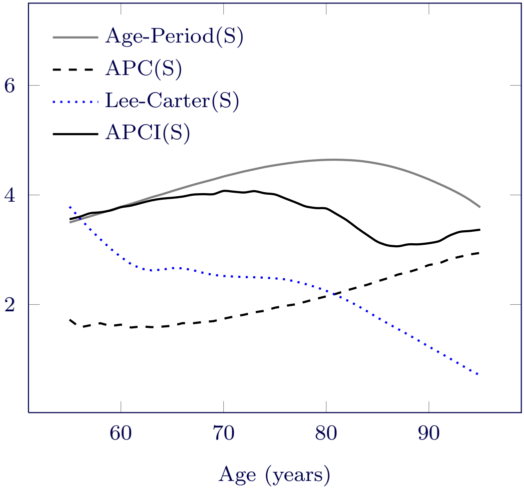
VaR99.5% capital-requirement percentages by age for four models. Source: Richards et al. [2017].



Q. Why do capital requirements reduce with age for Lee-Carter, but not with APCI?

A. κ_y is unmodulated by age in APCI model.

8 Conclusions

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8 Conclusions

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- APCI model is implementable as a fully stochastic model.
- APCI model shares features and drawbacks with Age-Period, APC and Lee-Carter models.
- Smoothing APCI $\hat{\alpha}_x$ and $\hat{\beta}_x$ seems sensible.
- Smoothing APCI $\hat{\kappa}_y$ and $\hat{\gamma}_{y-x}$ is not sensible.
- Currie [2013] algorithm makes fitting penalized, smoothed GLMs straightforward.

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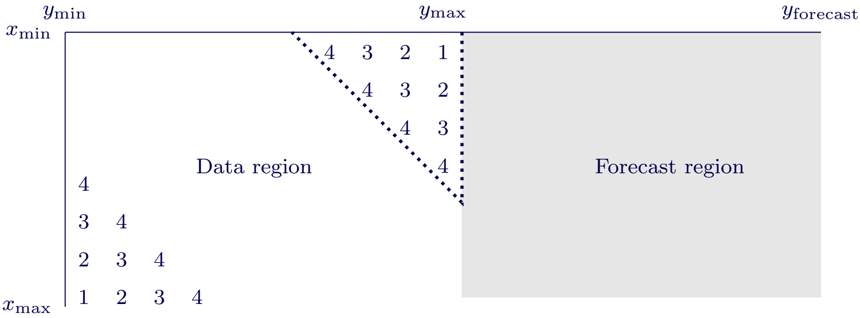
More on longevity risk at www.longevitas.co.uk

10 Constraints (again)

10 Corner cohorts

Number of observations for each cohort in the data region.



10 Constraints (again)

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- Both Continuous Mortality Investigation [2017] and Richards et al. [2017] avoid estimating "corner cohorts".
- This means not all constraints are required for identifiability.
- Continuous Mortality Investigation [2017] and Richards et al. [2017] both fit over-constrained APCI models.
- What impact does this have?

10 Constraints (again)

- Over-constrained models reduce the goodness-of-fit…

  …but can be used to impose desirable behaviour on parameters.

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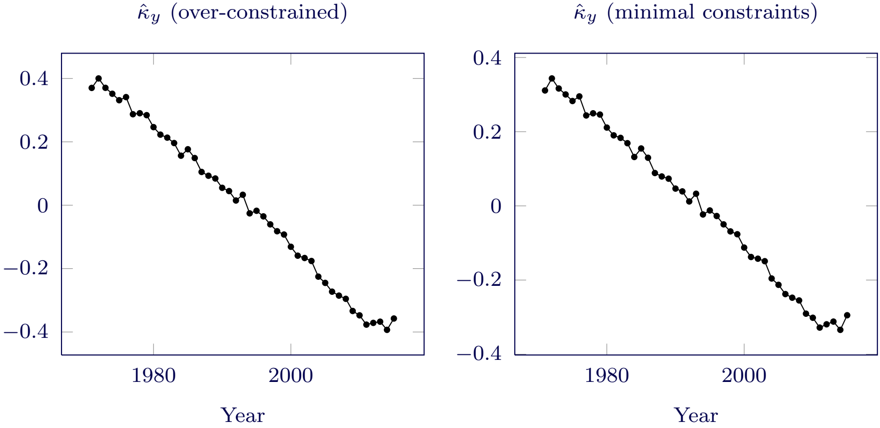
Parameter estimates $\hat{\kappa}_y$ APC(S) model



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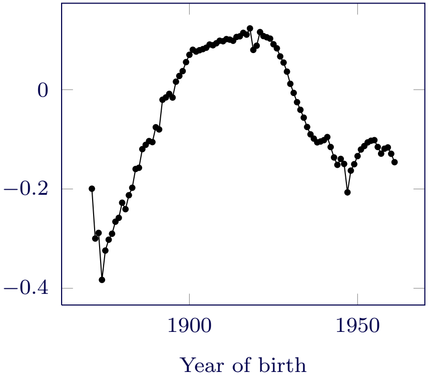
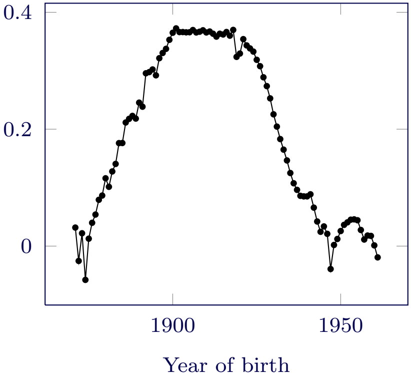
Parameter estimates $\hat{\gamma}_{y-x}$ APC(S) model

$\hat{\gamma}_{y-x}$ (over-constrained)



$\hat{\gamma}_{y-x}$ (minimal constraints)



10 APC model

- $\hat{\kappa}_y$ robust to over-constrained model.
- Values for $\hat{\gamma}_{y-x}$ differ, but shape similar.

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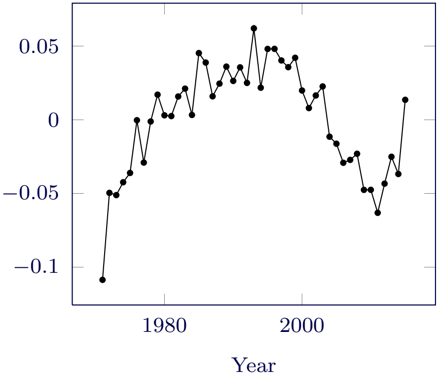
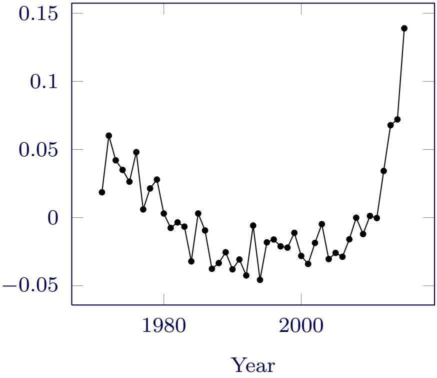
Parameter estimates $\hat{\kappa}_y$ APCI(S) model

$\hat{\kappa}_y$ (over-constrained)



$\hat{\kappa}_y$ (minimal constraints)



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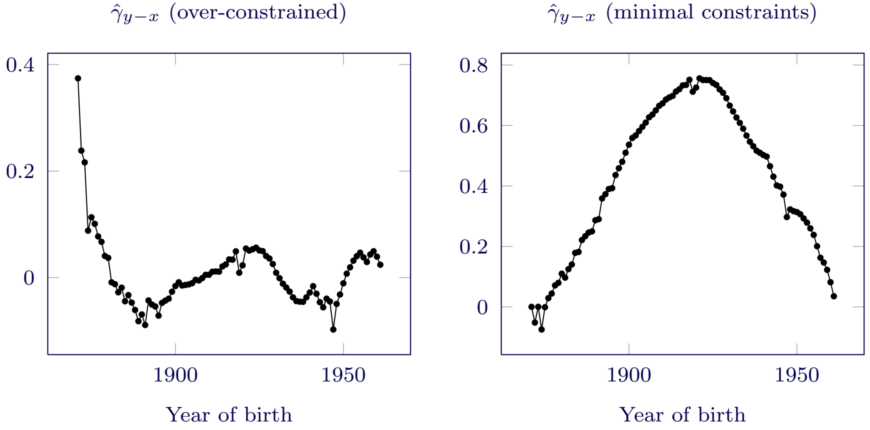
Parameter estimates $\hat{\gamma}_{y-x}$ APCI(S) model



10 APCI model

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- Neither $\hat{\kappa}_y$ nor $\hat{\gamma}_{y-x}$ robust to over-constrained model.
- κ_y in APCI model is a term which picks up left-over aspects of fit.
- $\hat{\gamma}_{y-x}$ changes radically depending on constraint choices.

  $\Rightarrow$ What are the implications for the CMI spreadsheet of using $\hat{\gamma}_{y-x}$ from APCI model?